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ON A COMPUTATIONALLY EFFICIENT MICROCOMPUTER KINEMATIC ANALYSIS OF THE BASIC LINKAGE MECHANISMS

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Abstract—Based on the algebraic correspondence between the displacement equations of the plane, spherical and skew four-bar linkages and the plane and skew slider-crank mechanisms, a highly compact, computationally efficient procedure has been developed for a microcomputer kinetmatic analysis of these linkages.

INTRODUCTION

The computer-aided kinematic analysis of linkage mechanisms can be implemented via large-scale, general-purpose codes (such as ADAMS, DADS, DRAM, IMP etc.) or special-purpose microcomputer codes [1]. The former typically utilize incremental numerical solutions of the loop-closure equations, while the latter are generally limited to single-loop mechanisms for which algebraic displacement equations can be formulated.

In connection with the latter group it has occurred to us that a single program applicable to many basic linkage mechanisms could be developed by utilizing the algebraic correspondence between their displacement equations. The following represents a development of these ideas.

THE ALGEBRAIC CORRESPONDENCE BETWEEN THE DISPLACEMENT EQUATIONS OF THE BASIC LINKAGE MECHANISMS

General observations

In his classic monograph [2], F. M. Dimentberg already found an algebraic correspondence between the displacement equations of the plane and spherical four-bar linkages. Considering the half-tangent form of these equations, Dimentberg showed that for any spherical four-bar linkage there exists a corresponding plane four-bar linkage such that for a given crank angle the half tangents of the output-link displacements are in constant proportion. In principle this would permit a kinematic analysis of both plane and spherical four-bar linkages by means of a single computer code, say, for the plane four-bar linkage, the transformation equations between the link lengths of the corresponding linkages being included in the code.

In the present investigation we have considered another type of correspondence between the displacement equations of five basic linkage mechanisms: the plane, spherical and skew four-bar linkages and the plane and skew slider cranks. The correspondence resides in the terms of their displacement equations expressed in half-tangent form. A generalized displacement equation is developed for all of the five mechanisms with velocities, as well as accelerations, obtainable by differentiation.

Algebraic development

The half-tangent form of the displacement equations of the basic linkages shown in Figs 1 and 2 are well known [2]. In order to clarify the algebraic correspondence between the displacement equations of the linkage mechanisms it is desirable to express the coefficients of the displacments equations as functions of the sums and differences of the four basic link lengths involved (Figs 1 and 2). In the case of the plane and skew four-bar linkages this involves factorization of terms which can be expressed as the differences of two squares, while in the case of the spherical four-bar linkage this involves conversion of the difference of two cosine terms into a product of two sine terms.

In Figs 1 and 2 the lengths of the fixed link, crank, coupler and output link are denoted by a, b, c or c^* and d, respectively. The angular positions (ϕ, ψ) of input and output link are defined by half tangents $t = \tan \frac{1}{2}\phi$ and $u = \tan \frac{1}{2}\psi$ or x, respectively.

Letting

$$\left.\begin{aligned} p_1 &= (1/2)\ (a - b + c - d) \\ p_2 &= (1/2)\ (a - b - c - d) \\ p_3 &= (1/2)\ (a + b + c - d) \\ p_4 &= (1/2)\ (a + b - c - d) \\ p_5 &= (1/2)\ (a - b + c + d) \\ p_6 &= (1/2)\ (a - b - c + d) \\ p_7 &= (1/2)\ (a + b + c + d) \\ p_8 &= (1/2)\ (a + b - c + d) \end{aligned}\right\} \quad (1)$$

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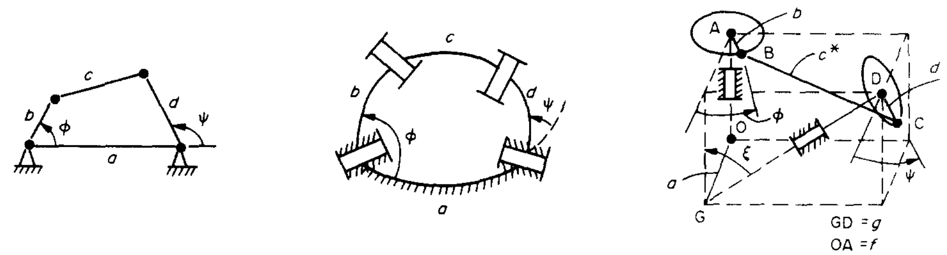


(a) Plane four-bar linkage

(b) Spherical four-bar linkage; dimensions are angular dimensions in radians or degrees

(c) Skew four-bar linkage: $c^2 = c^{*2} - (f^2 + g^2 - 2fg\cos\xi)$
f = OA
g = GD
$\xi = \angle(\mathrm{GD},\mathrm{OA})$

Fig. 1. The basic four-bar linkages.

the displacment equations of these linkages can be expressed in the following form:

$$A_1t^2 + A_2u^2 + A_3u^2t^2 + A_4ut + A_5 + A_6t(u^2+1) + (A_7 + A_{10})ut^2 + (A_7 + A_8)u + A_9t = 0 \quad (2)$$

$$\left.\begin{aligned}
A_1 &= \nabla_1 p_7 \nabla_1 p_8\\
A_2 &= \nabla_2 p_1 \nabla_2 p_2 + s\\
A_3 &= \nabla_3 p_3 \nabla_3 p_4 + s\\
A_4 &= (1-s)[-2\nabla_4 b \nabla_4 d] + \nabla_4 s\\
A_5 &= \nabla_5 p_5 \nabla_5 p_6\\
A_6 &= \nabla_6\\
A_7 &= \nabla_7\\
A_8 &= \nabla_8\\
A_9 &= \nabla_9\\
A_{10} &= \nabla_{10}
\end{aligned}\right\} \quad (3)$$

and where $s = 1$ for a slider-crank mechanism while $s = 0$ for a four-bar linkage. The operator, ∇_i $(i = 1, \ldots, 10)$ is defined in Table 1. By differentiation of the displacement equation (2) the derivatives defining the angular velocities and accelerations are readily obtained. For the four-bar linkages, for example, we find:

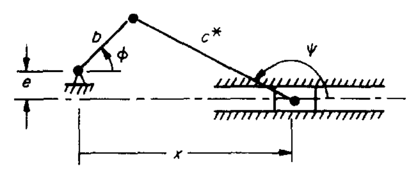
$$m_1 = \frac{d\psi}{d\phi} = -\frac{1+t^2}{1+u^2} \times \frac{A_1t + A_3tu^2 + \frac{1}{2}A_4u + \frac{1}{2}A_6(u^2+1) + (A_7 + A_{10})tu + \frac{1}{2}A_9}{A_2u + A_3t^2u + \frac{1}{2}A_4t + A_6ut + \frac{1}{2}(A_7 + A_{10})t^2 + \frac{1}{2}(A_7 + A_8)} \quad (4)$$

and

$$m_2 = \frac{d^2\psi}{d\phi^2} = \frac{-(Pm_1^2 + Qm_1 + R)}{S} \quad (5)$$

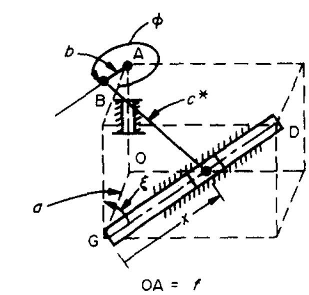
where

$$\left.\begin{aligned}
P &= (3u^2+1)(A_2 + A_3t^2 + A_6t)\\
&\quad + u[(A_7 + A_{10})t^2 + A_4t + A_7 + A_8]\\
Q &= 2(1+t^2)[2A_3tu + \tfrac{1}{2}A_4 + A_6u\\
&\quad + (A_7 + A_{10})t]\\
R &= \frac{1+t^2}{1+u^2}\{(3t^2+1)[A_1 + A_3u^2\\
&\quad + (A_7 + A_{10})u] + t[A_6(u^2+1)\\
&\quad + A_4u + A_9]\}\\
S &= 2[A_2u + A_3t^2u + \tfrac{1}{2}A_4t + A_6tu\\
&\quad + \tfrac{1}{2}(A_7 + A_{10})t^2 + \tfrac{1}{2}(A_7 + A_8)]
\end{aligned}\right\} \quad (6)$$



$c = \sqrt{c^{*2} - e^2}$

(a) Plane slider-crank



$c = \sqrt{c^{*2} - f^2}$

(b) Skew slider-crank

Fig. 2. The basic slider-crank mechanisms.

Table 1

i	Coefficient	Operator Δ_i for				
		Plane four-bar linkage	Spherical four-bar linkage	Skew four-bar linkage	Plane slider crank	Skew slider crank
1	A_1	1	sin	1	2	2
2	A_2	1	sin	1	0	0
3	A_3	1	sin	1	0	0
4	A_4	1	sin	$\sqrt{\cos\xi}$	0	$-4b\sin\xi$
5	A_5	1	sin	1	2	2
6	A_6	0	0	$-gb\sin\xi$	0	0
7	A_7	0	0	$fd\sin\xi$	0	0
8	A_8	0	0	0	$-2b$	$-2f\cos\xi$
9	A_9	0	0	0	$-4eb$	0
10	A_{10}	0	0	0	$2b$	$-2f\cos\xi$

and similarly for the slider cranks with

$$m_1' = \frac{d\chi}{d\phi} = \tfrac{1}{2}(1+u^2)m_1$$

and (7)

$$m_2' = \frac{d^2\chi}{d\phi^2} = \tfrac{1}{2}(1+u^2)\,(m_1^2u + m_2)$$

The computer program, written in BASIC for an IBM P/C, is given in the Appendix.

DISCUSSION

Figures 3–6 illustrate the displacements, velocities and accelerations of the output member of several linkage mechanisms according to the above program. The results were compared to those of a conventional kinematic analysis in order to verify accuracy.

The present approach could be extended to other linkages, if desired, the limitation being essentially the complexity of the mechanism. An alternative approach would be to program the kinematic anal-

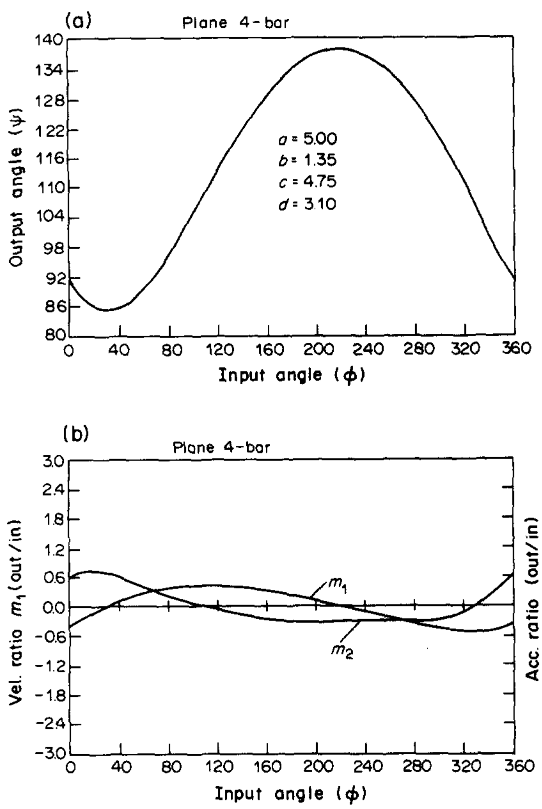


Fig. 3. Plane four-bar linkage (a) displacement analysis (b) velocity and acceleration analysis.

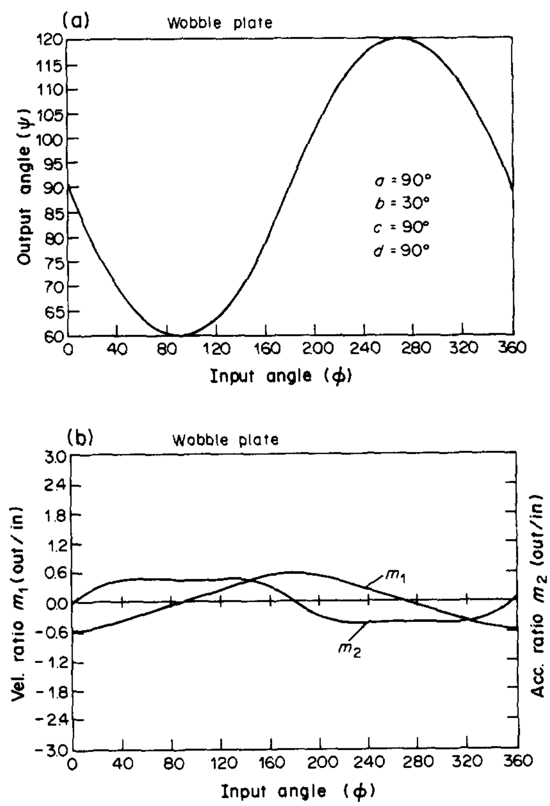


Fig. 4. Spherical wobble plate linkage (a) displacement analysis (b) velocity and acceleration analysis.

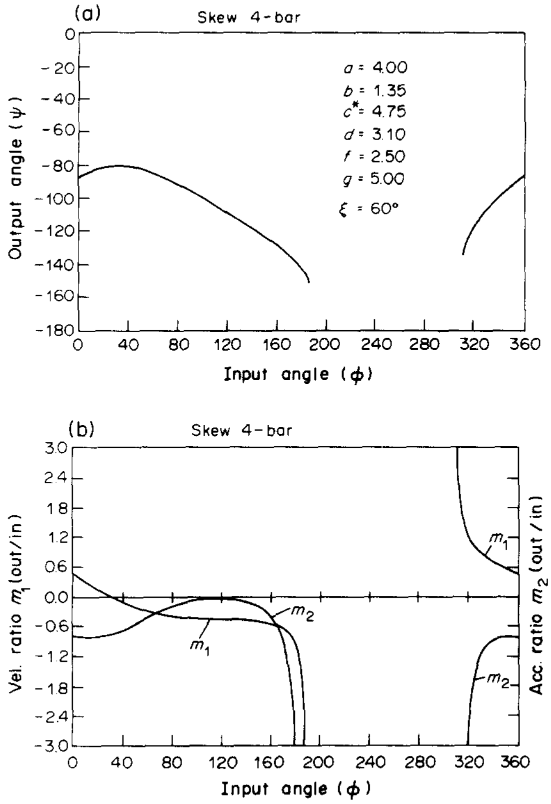


Fig. 5. Skew four-bar linkage (a) displacement analysis (b) velocity and acceleration analysis.

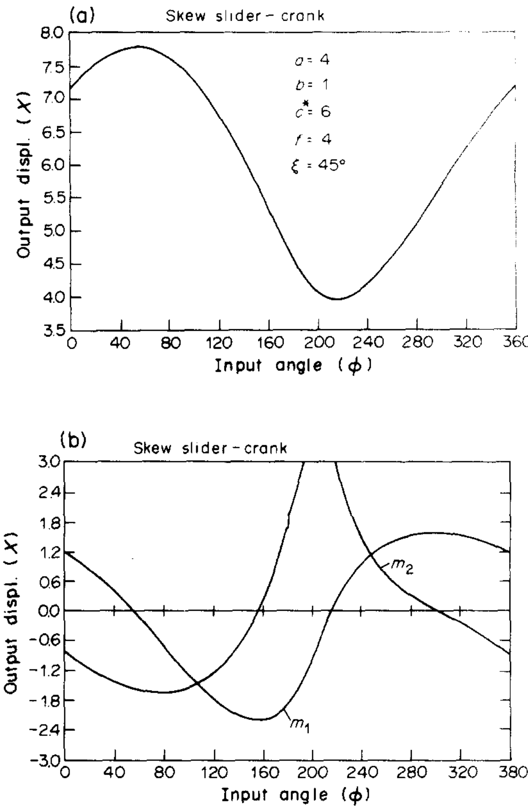


Fig. 6. Skew slider-crank (a) displacement analysis (b) velocity and acceleration analysis.

ysis code only for the most complex or parent linkage (in this case the skew four-bar) and to utilize the kinematic correspondence between the simpler linkages and the parent linkage. While this is feasible, the analysis may lead to indeterminacies for mechanisms with sliding pairs.

CONCLUSION

A simple, computationally efficient computer program has been developed for the kinematic analysis of five basic linkage mechanisms on a microcomputer. Extensions to other mechanisms are conceivable as well.

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REFERENCES

1. J. K. Krouse, Designing mechanisms on a personal computer. *Machine Design* 24 March, 94–99 (1983).
2. F. M. Dimentberg, *Determination of the Motions of Spatial Mechanisms*. Akad. Nauk, Moscow (1950) [Russian).

APPENDIX

```
10  '_______________________________________________________________
20  ':                                                              :
30  ':        Kinematics Analysis of Basic Linkage Mechanisms:      :
40  ':          1.  Plane Four-Bar Linkage.                         :
50  ':          2.  Spherical Four-Bar Linkage.                     :
60  ':          3.  Skew Four-Bar Linkage.                          :
70  ':          4.  Plane Slider-Crank.                             :
80  ':          5.  Skew Slider-Crank.                              :
90  ':______________________________________________________________:
100 KEY OFF
110 PI=3.141592654#:CIM=0
120 DIM DEL(7),M1(2),M11(2),M2(2),M22(2),U(2),PIM(8),P(8),PSI(2),PHIB(2)
130 CLS:LOCATE 6,25:PRINT "1. Plane Four Bar"
140 LOCATE 8,25:PRINT "2. Spherical Four Bar"
150 LOCATE 10,25:PRINT "3. Skew Four Bar"
160 LOCATE 12,25:PRINT "4. Plane Slider Crank"
```

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170 LOCATE 14,25:PRINT "5. Skew Slider Crank"
180 LOCATE 16,25:PRINT "6. Exit"
190 LOCATE 23,25:PRINT "Enter Option: _"
200 IN11$=INKEY$
210 IF IN11$="" THEN GOTO 200
220 IF VAL(IN11$)>=1 AND VAL(IN11$)<=5 THEN BARTYPE=VAL(IN11$):GOTO 250
230 IF IN11$="6" THEN GOTO 1360
240 SOUND 1500,1:GOTO 200
250 IF BARTYPE>=4 THEN S=1
260 CLS:INPUT "Please enter the input crank length: ",B:BT=B
270 INPUT "       enter the connecting rod length: ",C:CT=C
280 IF S=1 THEN GOTO 300
290 INPUT "       enter the ouput crank length: ",D:DT=D
300 IF BARTYPE <= 2 THEN INPUT "       enter the fixed link length: ",A:DIAG=A
310 IF BARTYPE=3 OR BARTYPE = 5 THEN INPUT "       enter the horizontal sector o
f the fixed link length: ",A
320 IF BARTYPE=4 THEN INPUT "       enter the center offset: ",E:GOTO 450
330 IF BARTYPE<>2 THEN GOTO 380
340 A=(A/180!)*PI:DIAG=A-(INT(A/PI)*PI)
350 B=(B/180!)*PI:BT=B-(INT(B/PI)*PI)
360 C=(C/180!)*PI:CT=C-(INT(C/PI)*PI)
370 D=(D/180!)*PI:DT=D-(INT(D/PI)*PI)
380 IF BARTYPE<> 3 AND BARTYPE<> 5 THEN GOTO 480
390 INPUT "       enter the vertical sector of the fixed link: ",F
400 INPUT "       enter the skew angle: ",KSI:KSI=(KSI/180!)*PI:DIAG=(A^2+(F*TAN
(KSI))^2)^.5
410 IF BARTYPE=5 THEN G=0:GOTO 430
420 G=F/COS(KSI)
430 C=(C^2-F^2-G^2+2*F*G*COS(KSI))
440 GOTO 460
450 C=(C^2-E^2)
460 IF C>=0 THEN C=C^.5:CIM=0
470 IF C<0 THEN CIM=1:C=(-C)^.5
480 IF BARTYPE <> 4 AND (DIAG>BT+CT+DT OR BT>DIAG+CT+DT OR CT>DIAG+BT+DT OR DT>D
IAG+BT+CT) THEN GOTO 500
490 GOTO 520
500 CLS:LOCATE 12,20:PRINT "Illegal linkage (impossible to assemble)":LOCATE 23,
25:PRINT "Hit any key to continue"
510 ASD$=INPUT$(1):GOTO 130
520 IF CIM=0 THEN FOR LOP=1 TO 8:PIM(LOP)=0:NEXT:GOTO 540
530 GOTO 590
540 P(1)=(A-B+C-D)/2!:P(2)=(A-B-C-D)/2!
550 P(3)=(A+B+C-D)/2!:P(4)=(A+B-C-D)/2!
560 P(5)=(A-B+C+D)/2!:P(6)=(A-B-C+D)/2!
570 P(7)=(A+B+C+D)/2!:P(8)=(A+B-C+D)/2!
580 GOTO 640
590 FOR LOP=1 TO 8:PIM(LOP)=((-1)^(LOP+1))*C/2!:NEXT
600 P(1)=(A-B-D)/2!:P(2)=P(1)
610 P(3)=(A+B-D)/2!:P(4)=P(3)
620 P(5)=(A-B+D)/2!:P(6)=P(5)
630 P(7)=(A+B+D)/2!:P(8)=P(7)
640 IF BARTYPE<>3 THEN GOTO 730
650 A1=(P(7)*P(8))-(PIM(7)*PIM(8))
660 B1=(P(1)*P(2))-(PIM(1)*PIM(2))
670 C1=(P(3)*P(4))-(PIM(3)*PIM(4))
680 D1=-2*COS(KSI)*B*D
690 E1=(P(5)*P(6))-(PIM(5)*PIM(6))
700 F1=-G*B*SIN(KSI)
710 G1=F*D*SIN(KSI):H1=0:I1=0:J1=0
720 GOTO 800
730 IF BARTYPE < 4 THEN GOTO 800
740 A1=4*((P(7)*P(8))-(PIM(7)*PIM(8))):B1=S:C1=S
750 D1=-4*B*SIN(KSI):IF BARTYPE=4 THEN D1=0
760 E1=4*((P(5)*P(6))-(PIM(5)*PIM(6)))
770 F1=0:G1=0:I1=0:IF BARTYPE=4 THEN I1=-4*E*B
780 H1=-2*F*COS(KSI):IF BARTYPE=4 THEN H1=-2*B
790 J1=-2*F*COS(KSI):IF BARTYPE=4 THEN J1=2*B
800 IF BARTYPE<>2 THEN GOTO 870
810 A1=SIN(P(7))*SIN(P(8))
820 B1=SIN(P(1))*SIN(P(2))
830 C1=SIN(P(3))*SIN(P(4))
840 D1=-2*SIN(B)*SIN(D)
850 E1=SIN(P(5))*SIN(P(6))
860 F1=0:G1=0:H1=0:I1=0:J1=0
870 IF BARTYPE <>1 THEN GOTO 940
880 A1=P(7)*P(8)
890 B1=P(1)*P(2)
900 C1=P(3)*P(4)
```

```
910 D1=-2*B*D
920 E1=P(5)*P(6)
930 F1=0:G1=0:H1=0:I1=0:J1=0
940 INPUT "Enter the first bound of the input angle: ",PHIB(1)
950 INPUT "Enter the second bound of the input angle: ",PHIB(2)
960 INPUT "Enter increment: ",INC:INC=(INC/180!)*PI
970 PRINT "Which solution do you need?  1. (A+B)    2. (A-B)"
980 ANS$=INPUT$(1):SOL=VAL(ANS$)
990 IF SOL<1 OR SOL>2 THEN SOUND 1500,1:GOTO 980
1000 PHIB(1)=(PHIB(1)/180!)*PI
1010 PHIB(2)=(PHIB(2)/180!)*PI
1020 OPEN "O",#1,"ANG.DAT"
1030 OPEN "O",#2,"VEL.DAT"
1040 OPEN "O",#3,"ACC.DAT"
1050 FOR PHI=PHIB(1) TO PHIB(2) STEP INC
1060 T=TAN(.5*PHI)
1070 U11=-(D1*T+(G1+J1)*T^2+(G1+H1))/(2*(B1+C1*(T^2)+F1*T))
1080 IF ((D1*T+(G1+J1)*T^2+(G1+H1))^2)<4*(B1+C1*(T^2)+F1*T)*(A1*(T^2)+E1+F1*T+I1
*T) THEN U11=0:U12=0:UIM=1:GOTO 1100
1090 U12=((((D1*T+(G1+J1)*T^2+(G1+H1))^2)-4*(B1+C1*(T^2)+F1*T)*(A1*(T^2)+E1+F1*T
+I1*T))^.5)/(2*(B1+C1*(T^2)+F1*T))
1100 U(1)=U11+U12
1110 U(2)=U11-U12
1120 PSI(1)=U(1):PSI(2)=U(2)
1130 IF S=0 THEN PSI(1)=(2*ATN(U(1))/PI)*180!
1140 IF S=0 THEN PSI(2)=(2*ATN(U(2))/PI)*180!
1150 WRITE #1,((PHI/PI)*180!),PSI(SOL)
1160 FOR LOP=1 TO 2
1170 M1(LOP)=-(1+T^2)*(A1*T+C1*T*U(LOP)^2+.5*D1*U(LOP)+.5*F1*(U(LOP)^2+1)+(G1+J1
)*T*U(LOP)+.5*I1)/((1+U(LOP)^2)*(B1*U(LOP)+C1*U(LOP)*T^2+.5*D1*T+F1*U(LOP)*T+.5*
(G1+J1)*T^2+.5*(G1+H1)))
1172 M11(LOP)=M1(LOP)
1175 IF S=1 THEN M11(LOP)=(1+U(LOP)^2)*M1(LOP)/2!
1180 IF UIM=1 THEN M11(LOP)=0:M1(LOP)=0
1200 IF M11(LOP)>3 THEN M11(LOP)=3
1210 IF M11(LOP)<-3 THEN M11(LOP)=-3
1220 P1=(3*U(LOP)^2+1)*(B1+C1*T^2+F1*T)+U(LOP)*((G1+J1)*T^2+D1*T+G1+H1)
1230 Q1=2*(1+T^2)*(2*C1*T*U(LOP)+.5*D1+F1*U(LOP)+(G1+J1)*T)
1240 R1=((1+T^2)/(1+U(LOP)^2))*((3*T^2+1)*(A1+C1*U(LOP)^2+(G1+J1)*U(LOP))+T*(F1*
(U(LOP)^2+1)+D1*U(LOP)+I1))
1250 S1=2*(B1*U(LOP)+C1*U(LOP)*T^2+.5*D1*T+F1*U(LOP)*T+.5*(G1+J1)*T^2+.5*(G1+H1)
)
1260 M2(LOP)=-(P1*M1(LOP)^2+Q1*M1(LOP)+R1)/S1
1262 M22(LOP)=M2(LOP)
1265 IF S=1 THEN M22(LOP)=((M1(LOP)^2)*U(LOP)+M2(LOP))*(1+U(LOP)^2)/2!
1270 IF UIM=1 THEN M2(LOP)=0:M22(LOP)=0
1290 IF M22(LOP)>3 THEN M22(LOP)=3
1300 IF M22(LOP)<-3 THEN M22(LOP)=-3
1310 NEXT LOP
1320 WRITE #2,((PHI/PI)*180!),M11(SOL)
1330 WRITE #3,((PHI/PI)*180!),M22(SOL)
1340 UIM=0:NEXT PHI
1350 CLOSE
1360 END
```

UBER EIN RECHNERISCH VORTEILHAFTES MICRO-COMPUTER RECHENPROGRAMM FÜR DIE KINEMATISCHE ANALYSE DER FUNDAMENTALEN GELENKMECHANISMEN

Zusammenfassung—Aud grund einer algebraischen Korrespondenz zwischen den Bewegungsgleichungen der ebenen, sphärischen und räumlichen Gelenkvierecke, sowie der ebenen und räumlichen Schubkurbeln, wird ein rechnerisch vorteilhafte Methode entwickelt fur eine microcomputer kinematische Analyse dieser Getriebe.