Updating Procedures for Iterative Learning Control in Hilbert Space

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Abstract

The research results presented here are extensions of previous work on Iterative Learning Control (ILC) using projection-based update techniques. These updates were mainly developed for Quasi-Newton optimization and also for solving systems of simultaneous equations. Recently, such updates have been used by the authors to estimate the dynamics of a control system which is asked to perform a periodic task using a discrete formulation. In addition, recently, preliminary extensions of these updating techniques to continuous systems through formulation in Hilbert Space have been proposed. This paper provides a formulation of the Iterative Learning Control problem in Hilbert Space with a convergence proof of the proposed solution based on Broyden's update.

1 Introduction

Iterative Learning Control (ILC) refers to schemes which take advantage of the extra information available in controlling systems which are asked to perform a periodic task. Specifically, in Iterative Learning Control, the initial conditions of the controlled system are reset to the same value after each period. Early formulations of ILC, tried to re-use ideas developed in the classical control community such as the Proportional-Derivative Control scheme [1] or the the Integrator [17]. Later, some adaptive-control ideas started to be used and formulated to fit the case with a periodic trajectory and to take advantage of the extra information made available by the periodicity. Examples of such systems are given in [15] which uses the well-known self-tuning regulator idea and [2, 4, 16, 6] which are based on parameter estimation using updating techniques.

References [4, 6, 16] have presented a formulation of the Learning Control problem using a discrete linear time-dependent approximation to the system performing multiple periods of a periodic task. The initial conditions are reset after each period. [2], has presented a learning control scheme for continuous-time processes using definitions in Hilbert Space. The current paper brings together these two formulations and derives an Iterative Learning Control scheme for continuous systems using the Broyden update. A proof of convergence is provided for this case and an application to dynamical systems is proposed and formulated.

2 Problem Formulation

The following presents the formulation of the general Iterative Learning Control Problem.

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Let a system be described by an operator $F : U \to Y$, where U and Y are Hilbert spaces with inner products $< \cdot, \cdot >_u$ and $< \cdot, \cdot >_y$, respectively. We assume that the system is forced to work periodically, that is during each cycle we have

$$y_k = F(u_k), \quad k = 0, 1, ...,$$

where $u_k \in U$ and $y_k \in Y$ are a control and an observable output of the system at the k-th cycle, respectively. Note that the above setting is quite general: the operator F can represent either a continuous-time or a discrete-time plant. Let y_d be a desired behavior of the system. Our aim is to find control u_d which is a solution of the operator equation

$$y_d = F(u). \tag{1}$$

However, we do not know the operator F exactly. Since we are dealing with a periodic system, it allows us to use the information obtained from the previous cycles to improve the performance of the system at the next cycle as well as to update our knowledge about operator F. The latter remark prompts us to use some iterative numerical methods for the solution of the operator equation (1) on-line. In this paper, we will apply operator updating methods, which are known to be quite efficient in numerical procedures. The authors' second work, which deals with updating the operator as well as the control input, was originally done by the second and third authors in [16] and later with some extensions was published in [6] and [4]. This work covered a whole array of numerical techniques for performing this update with a comparison of those techniques. It was, however, applied to a discrete system. Here, we will extend the results and formulation of [4] and [6] in somewhat the same way as [2] had addressed the continuous-time problem in which it used a fixed estimate of the system operator.

Note that we form the control input for the next $(k+1)$-st cycle as follows:

$$u_{k+1} = u_k - P_k^{-1} e_k, \quad k = 0, 1, ..., \tag{2}$$

where $e_k = y_k - y_d$ is an error at the k-th cycle and $P_k \simeq F'(u_k)$ is some approximation of the linear part of the system near the trajectory induced by control u_k. We will also refer to P_k^{-1} as the learning operator. As it was mentioned before, [2] proposed a quasi-Newton ILC method where the structure of the learning operator once chosen in the beginning is not modified during the learning procedure. The purpose of the present paper is to analyze learning procedures where newly arrived information is used to update not only the control input but also the learning operator. [4] shows the convergence of such algorithms for a discrete-time systems. Here, we prove convergence for the case of non-linear system operator F acting on a general Hilbert space. In addition, we propose to use ILC based on the generalized secant method [4], which takes into account the information from several preceding iterations. Finally, we would like to note that quasi-Newton ILC methods can be considered as numerical optimization methods for some auxiliary mathematical programs [12, 13, 5, 16, 6], where Hessian matrices are updated in a similar fashion.

3 Methods for updating of the System Iterator

In this section we discuss updating procedures for iterative learning control based on projection-type operator-updating methods. Suppose we apply control u_{k+1} calculated as in (2) to the system and observe the output y_{k+1}. If the system were linear and P_k were chosen as the operator of the system, then the equation $P_k \Delta u = y_{k+1} - y_k$ would be satisfied. However, we neither have a linear system nor we know exactly the linear part of the system. Therefore, we update the approximation of the linear part P_k in such a way that the equation $P_{k+1} \Delta u = y_{k+1} - y_k$ is satisfied. In addition, we require changes in the approximation operator to be as small as possible. The above consideration leads to the next convex programming formulation

$$P_{k+1} = \arg\min_{P} ||P - P_k|| \tag{3}$$

subject to

$$P\Delta u = y_{k+1} - y_k.$$

or in terms of the error, e_k,

$$P\Delta u = e_{k+1} - e_k.$$

This optimization problem has a solution for any norm in the following form [10, 5]:

$$P_{k+1} = P_k + \frac{1}{< \Delta u_k, \Delta u_k >}(e_{k+1} - e_k - P\Delta u_k) \otimes \Delta u_k \tag{4}$$

The above formula was first obtained by Broyden [7] and we will refer to it as Broyden's update. It is a special case of the more general update given by Barnes [3] and used by [4, 16] in application to ILC.

Remark 1 *Note that the expression (4) provides a solution in the case of any norm used in (3). In general the solution can be non-unique. However, the above formula provides an optimal solution with good properties. This can be easily demonstrated in the finite-dimensional case. Namely, if in the case of finite dimensions one applies the Frobenious norm, which takes into account the values of all elements of matrix P, then (4) is a unique solution to the optimization problem (3).*

Using the standard technique from the theory of quasi-Newton methods [8, 9, 10, 11, 14], we can prove the following result.

Theorem 1 *Let $F : U \to Y$ be continuously differentiable in $S(u_d, r) = \{u \in U \mid ||u - u_d|| < r\}$, where u_d is a solution of the operator equation (1) and $r > 0$. Let $q \in (0, 1)$ and the following conditions hold.*

1) $||F'(u_d)|| \leq \beta,$
2) $||F'(u_1) - F'(u_2)|| \leq \gamma ||u_1 - u_2||,$
   $u_1, u_2 \in S(u_d, r),$
3) $5\beta\delta_q \leq q,$
4) $\dfrac{\gamma\varepsilon_q}{1 - q} \leq \delta_q$

Then, if $u_0 \in S(u_d, \varepsilon_q)$ and $||P_0 - F'(u_d)|| \leq \delta_q$, the sequence $\{u_k\}_{k=0}^{\infty}$ generated by ILC method (2) with updating procedure (4) is well defined and converges to u_d. The convergence is at least linear with rate q, that is

$$||u_{k+1} - u_d|| \leq q||u_k - u_d||, \quad k = 0, 1, ... \tag{5}$$

PROOF: We show by induction on k that (5) holds. We omit the proof for $k = 0$, since it is identical to the proof of the induction step. Suppose that (5) is true for $k = 0, ..., l - 1$ and let us demonstrate that it is also true for $k = l$.

First, we prove some auxiliary results. The following estimation is true for Broyden's update if one uses the L_2-norm [8, 10, 11].

$$||P_k - F'(u_d)|| \le ||P_{k-1} - F'(u_d)|| + \frac{\gamma}{2}(||u_k - u_d|| + ||u_{k-1} - u_d||) \tag{6}$$

By iterating (6), we obtain

$$\begin{aligned} ||P_l - F'(u_d)|| &\le ||P_0 - F'(u_d)|| + \frac{\gamma}{2}||u_0 - u_d \quad || \\ &\quad + \gamma \sum_{i=1}^{l-1} ||u_i - u_d|| + \frac{\gamma}{2}||u_l - u_d \quad || \\ &\le ||P_0 - F'(u_d)|| + \gamma \sum_{i=0}^{l} ||u_i - u_d \quad || \end{aligned} \tag{7}$$

Then, using the induction hypothesis and Condition 4, we get

$$||P_l - F'(u_d)|| \le ||P_0 - F'(u_d)|| + \gamma\varepsilon_q \sum_{i=0}^{l} q^i \le \delta_q + \frac{\gamma\varepsilon_q}{1-q} \le 2\delta_q, \tag{8}$$

that is

$$||P_l - F'(u_d)|| \le 2\delta_q. \tag{9}$$

Next we show that the operator P_l^{-1} is well defined. Namely, using the Banach perturbation theorem [10, 14], the above inequality (9) and Condition 3, we can write

$$||P_l^{-1}|| \le \frac{||F'(u_d)^{-1}||}{1 - ||F'(u_d)^{-1}||||P_l - F(u_d)||} \le \frac{\beta}{1 - \beta 2\delta_q} \le \frac{\beta}{1 - 2/5} = \frac{5\beta}{3}. \tag{10}$$

Now we are able to prove the induction step.

$$\begin{aligned} ||u_{l+1} - u_d|| &\le ||u_l - u_d - P_k^{-1} z_l|| \\ &\le ||u_l - u_d - P_l^{-1}(F(u_l) - F(u_d))|| \\ &\le ||P_l^{-1}[F(u_d) - F(u_l) \\ &\quad -P_l(u_d - u_l)]|| \end{aligned} \tag{11}$$

The expression in the square brackets of (11) can be rewritten as follows:

$$\begin{aligned} F(u_d) - F(u_l) &- P_l(u_d - u_l) = \\ = \int_0^1 F'(u_l &+ t(u_d - u_l))(u_d - u_l)dt \\ &- P_l(u_d - u_l) \\ = \int_0^1 [F'(u_l &+ t(u_d - u_l)) - F'(u_l) \\ &+ F'(u_l) - P_l] \quad (u_d - u_l)dt \end{aligned} \tag{12}$$

Substituting the above expression back into inequality (11), we obtain

$$
\begin{aligned}
||u_{l+1} - u_d|| &\leq ||P_l^{-1}|| || \int_0^1 [F'(u_l + t(u_d - u_l)) - F'(u_l) + F'(u_l) - P_l](u_d - u_l)dt|| \\
&\leq ||P_l^{-1}|| (\int_0^1 ||F'(u_l + t(u_d - u_l)) - F'(u_l)|| \, ||u_d - u_l||dt + \\
&\quad ||F'(u_l) - P_l|| ||u_d - u_l|| \int_0^1 dt) \\
&\leq ||P_l^{-1}|| (\int_0^1 \gamma ||t(u_d - u_l)|| ||u_d - u_l||dt + ||F'(u_l) - P_l|| ||u_d - u_l||) \\
&\leq ||P_l^{-1}|| (\frac{\gamma}{2} ||u_l - u_d)|| + ||F'(u_l) - P_l||) ||u_l - u_d||
\end{aligned}
$$

Condition 2 was used in the above calculations. From auxiliary estimates (9) and (10) it follows that

$$||u_{l+1} - u_d|| \leq \frac{5\beta}{3} (\frac{\gamma \varepsilon_q}{2} + 2\delta_q) ||u_l - u_d||$$

Note that $\frac{\gamma \varepsilon_q}{2} < \frac{\gamma \varepsilon_q}{1-q} \leq \delta_q$, hence

$$||u_{l+1} - u_d|| \leq \frac{5\beta}{3} 3\delta_q ||u_l - u_d||$$

Finally, using Condition 3, we obtain

$$||u_{l+1} - u_d|| \leq 5\beta\delta_q ||u_l - u_d|| \leq q ||u_l - u_d||.$$

This completes the proof.

$\square$

Remark 2 *Moreover, in the finite-dimensional case, one can prove super-linear convergence of the ILC method with Broyden's update. The super-linear convergence [10] means that* $\|e_{k+1}\| \leq c_k \|e_k\|$ *with* $c_k \to 0$ *as* $k \to \infty$. *This fact shows that projection-type ILC methods with updating procedures outperform the method of [2] with the fixed structure of the learning operator which possess only linear convergence.*

Note that by using the Sherman-Morrison formula, one can update the inverse operator P_k^{-1} rather than P_k. Namely, we have

$$P_{k+1}^{-1} = P_k^{-1} + \frac{1}{< \Delta u_k, P_k^{-1}(y_{k+1} - y_k) >} [\Delta u_k - P_k^{-1}(y_{k+1} - y_k)] \otimes \Delta u_k P_k^{-1} \tag{13}$$

or, equivalently,

$$P_{k+1}^{-1} = P_k^{-1} + \frac{1}{< \Delta u_k, P_k^{-1} \Delta y_k >} [\Delta u_k - P_k^{-1} \Delta y_k] \otimes \Delta u_k P_k^{-1} \tag{14}$$

with $\Delta y_k := y_{k+1} - y_k$.

4 Generalized Secant in Hilbert Space

In this section, the Generalized Secant solution to the ILC problem is formulated in Hilbert space using the techniques discussed so far. This was first presented for a discrete linear time-dependent approximation of a learning control system in [16] and later refined in [6] and [4].

The parameter update formula for the generalized secant solution is given by,

$$P_{k+1} = P_k + \frac{1}{< z_k, \Delta u_k >}(e_{k+1} - e_k - P\Delta u_k) \otimes z_k \tag{15}$$

where z_k are some projection vectors. These vectors are picked to be orthogonal to be able to speed up the convergence of the system operator and hence the control strategy. It has been shown in [4] that for the discrete case, the system will converge in only N iterations where N is the rank of the operator matrix for the discrete case.

Note that the Broyden update given by 4 is a special case of the so called generalized secant method [3, 4]. In the Broyden update, z_k is set to be equal to the change in the control action from one period to the next, namely, Δu_k.

For the discrete case, [4] shows that in order to have quicker convergence, it is beneficial to apply the change in the control input from one period to the next in small magnitudes, but retain the direction given by the algorithm. This amounts to a parameter estimation with small perturbations from the original trajectory. A confidence criterion is also given in [4] for the discrete case which dictates the readiness for taking larger corrective steps in the control input for the next period.

In future publications, the convergence of the generalized secant method will be analyzed in the context of general Hilbert spaces and the results compared to those obtained earlier. Also, the confidence criterion is explored for the continuous case.

5 Application to dynamical systems

In this section we demonstrate an application of the proposed solution to the Iterative Learning Control problem using Broyden's update to the learning control of Lagrangian dynamical systems. A Lagrangian dynamical system is given by the following vector differential equation

$$\begin{aligned} & A(q)\ddot{q} + b(q, \dot{q}) = u, \\ & q(t)|_{t=0} = q(0), \;\; \dot{q}(t)|_{t=0} = \dot{q}(0), \end{aligned} \tag{16}$$

where $q \in \mathbb{R}^n$ is a vector of generalized coordinates and $u \in \mathbb{R}^n$ is a vector of generalized forces. We need to tune the system so that it can track repeatedly the trajectory $q_d(t), t \in [0, T]$. Denote

$$e_k(t) := q_k(t) - q_d(t), t \in [0, T], k = 0, 1, ...$$

Then for the first learning step we can apply the standard procedure (see e.g. [2, 18, 19]):

$$u_1(t) = u_0(t) - (\hat{A}\ddot{e}_0(t) + \hat{B}\dot{e}_0(t) + \hat{C}e_0(t)), \tag{17}$$

where $\hat{A}\ddot{y} + \hat{B}\dot{y} + \hat{C}y = u$ is some linear approximation of the system (16). In this case,

$$P_0^{-1} = \hat{A}\frac{d^2}{dt^2} + \hat{B}\frac{d}{dt} + \hat{C}I.$$

Then, applying the inverse updating formula (14), we obtain

$$\begin{aligned} u_2(t) &= u_1(t) - (\hat{A}\ddot{e}_1(t) + \hat{B}\dot{e}_1(t) + \hat{C}e_1(t)) \\ &- \frac{\int_0^T \Delta u_0(\tau)[\hat{A}\ddot{e}_1(\tau) + \hat{B}\dot{e}_1(\tau) + \hat{C}e_1(\tau)]d\tau}{\int_0^T \Delta u_0(\tau)[\hat{A}\Delta\ddot{y}_0(\tau) + \hat{B}\Delta\dot{y}_0(\tau) + \hat{C}\Delta y_0(\tau)]d\tau} \\ &\quad \Big(\Delta u_0(t) - [\hat{A}\Delta\ddot{y}_0(t) + \hat{B}\Delta\dot{y}_0(t) + \hat{C}\Delta y_0(t)]\Big) \end{aligned} \tag{18}$$

Proceeding in a similar way, one can easily define the general recursive procedure.

6 Conclusion

This paper has formulated the general ILC problem using Broyden's update in continuous-time by representing the problem in Hilbert space. An application has also been proposed for the use of this control scheme.

References [4] and [16] have shown that on discrete systems, a Generalized Secant method, in which the projection dimensions are not necessarily co-incident with the change in the control input, converge in a finite number of steps. The authors are currently working on extending the Hilbert space formulation and convergence proof to the Generalized Secant update. Results of this research will be made available in future publications.

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